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ON A CLASS OF ISOMETRIC SUBGRAPHS OF A GRAPH

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In a graph G, which has a loop at every vertex, a connected subgraph $H=(V(H), E(H))$ is a retract if, for any $a, b\in V(H)$ and for any paths P, Q in G, both joining a to b, and satisfying $|Q|\geqq |P|$, then $P\subseteqq V(H)$ whenever $Q\subseteqq V(H)$. As such subgraphs can be described by a closure operator we are led to the investigation of the corresponding complete lattice of "closed" subgraphs. For example, in this complete lattice every element is the infimum of an irredundant family of infimum irreducible elements.

Introduction

This paper is inspired by the problem of characterizing the retracts of a graph [3—6].

For a graph G let $V(G)$ denote its vertex set and $E(G)\subseteqq V(G)\times V(G)$ its edge set. A subgraph H of G is a *retract* if there is an edge-preserving map f of $V(G)$ onto $V(H)$ satisfying $f(v)=v$ for each $v\in V(H)$. Retracts preserve certain of the familiar invariants of a graph. For instance, if G is an undirected graph with no loops and with no multiple edges then its chromatic number is preserved by every retract. Accordingly, K_2 is a retract of every two-chromatic graph. However, the classification of all minimal graphs [graphs without proper retracts] even with chromatic number three remains virtually unexamined. Such structure-preserving features of retracts invests the concept with importance.

Retracts in irreflexive graphs have been investigated in [3, 4, 5]. Quite independently, we have considered retraction problems for reflexive graphs in [6]. (An undirected graph G is *reflexive* if it has a loop at every vertex, but no multiple edges: its edge set $E(G)$ corresponds to a reflexive and symmetric binary relation on $V(G)$.) Hell's work in the irreflexive case centered on the problem of absolute retracts, that is, graphs which are retracts of any graph (in a given class of graphs) in which they can be isometrically embedded. Characterizations of absolute retracts in bipartite graphs are given in [3, 5], and in planar graphs in [4]. We have shown that, in a reflexive graph, every cycle of minimum order as well as every isometric tree is a retract. (In

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fact, a bipartite version of this result, alluded to in [6], provides a common ground with Hell's work, cf. [3].) Such facts, in turn, can have a bearing on the fixed point problem for partially ordered sets (since the comparability graphs of partially ordered sets are reflexive) [3]. Indeed, such facts may even have a bearing on a matter of concrete interest. Consider this: a set of factories [vertices] tied together by certain transportation links [edges] is to be rationalized by decreasing the number of factories and augmenting the productivity of others, while maintaining the diversity of production and especially the required transportation links. In other words, find an appropriate retract of the corresponding reflexive graph.

Let G be a graph. A *path* of G is a sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ of distinct vertices of G satisfying $(a_i, a_{i+1}) \in E(G)$ for each $i=1, 2, \ldots,$; the *length* $l(P)$ of a finite path $P=(a_0, a_1, a_2, \ldots, a_n)$ is n, where $n \geqq 1$. In this case we also say that P *joins* a_0 to a_n. If, furthermore, $n \geqq 2$ and $a_n = a_0$ then we call P a *cycle* (of length n). (Where convenient we shall also consider P as a subset of $V(G)$.) For $a, b \in V(G)$, the *distance* $d_G(a, b)$ between a and b in G is the least length (if it exists) of a path joining a to b. G is *connected* if, for every $a, b \in V(G)$ there is a finite path joining a to b.

Let H be a subgraph of a reflexive graph G. If H is a retract then H must be *isometric* in G $[d_H(a, b) = d_G(a, b)$ for each $a, b \in V(H)]$. This is not, however, a necessary condition (see Figure 1).

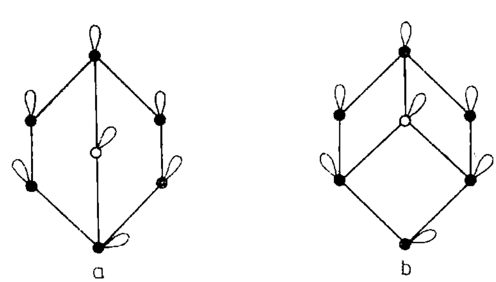


Fig. 1. In each graph the subgraph with shaded vertex set is not a retract

In a sense, the starting point for our current study lies in the following sufficient condition for a subgraph to be a retract of a reflexive graph (see Proposition 9):

> *in a reflexive graph G a connected subgraph H is a retract if, for any $a, b \in V(H)$ and for any paths P, Q both joining a to b, and satisfying $l(Q) \geqq l(P)$, then $P \subseteq V(H)$ whenever $Q \subseteq V(H)$.*

As such subgraphs can be described by a closure operator we are led naturally to their classification by investigating the order-theoretic characteristics of the corresponding lattice of "closed" subgraphs and especially in the case that the underlying graph is infinite.

Closed Subgraphs

We shall first dispense with some remaining items of terminology and notation.

For our purposes a graph G is an ordered pair $(V(G), E(G))$ whose vertex set $V(G)$ may be infinite and whose edge set $E(G)$ is a symmetric binary relation on $V(G)$. (It is convenient to suppress the requirement of a loop at every vertex unless and until we consider the matter of retracts.) A *subgraph* H of G is a graph satisfying $V(H) \subseteq V(G)$ and whose edges $E(H)$ are precisely those edges of G with both endpoints in $V(H)$ [induced subgraph]. In this way every subset U of $V(G)$ determines a subgraph of G; we denote it by $[U]$.

A graph G can be decomposed into its *components* (maximal connected subgraphs) $(G_\alpha | \alpha \in I)$ in each of which there is a metric d_{G_α} defined on $V(G_\alpha) \times V(G_\alpha)$ to $\mathbf{N}$ by

$$d_{G_\alpha}(a, b) = \min\big(l(P) | a, b \in P \text{ and } P \text{ is a path in } G_\alpha\big).$$

For our purposes another "distance" function is needed. For a subgraph H of G we define a function D_H of $V(H) \times V(H)$ to $\mathbf{N} \cup \{\infty\}$ as follows: if $a, b \in V(H)$ are in different components of G, or $a=b$, then

$$D_H(a, b) = 0;$$

if $a \neq b$ and a, b are in the same component of G, and if there is a longest path $P_{a,b}$ in H joining a to b, then

$$D_H(a, b) = \max\big(d_G(a, b),\ l(P_{a,b})\big)$$

(that is, $D_H(a, b) = l(P_{a,b})$, unless a, b are in different components of H, in which case $D_H(a, b) = d_G(a, b)$); if $a \neq b$ and a, b are in the same component of G, but H contains arbitrarily long paths joining a and b, then

$$D_H(a, b) = \infty$$

(see Fig. 2).

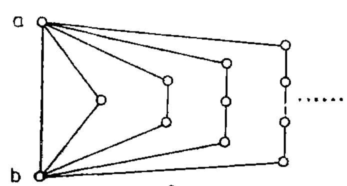


Fig. 2

In anticipation of our upcoming remarks we call a subgraph H, of a graph G, *enclosed* if, for each $a \neq b$ in $V(H)$ and, for each path P of G joining a to b, such that

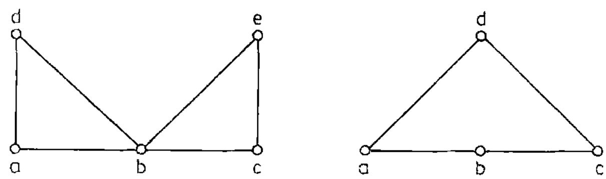
$$P \cap V(H) = \{a, b\}$$

then

$$l(P) > D_H(a, b).$$

Lemma 1. *Let H be an enclosed subgraph of a graph G and let $a, b \in V(H)$. If a, b belong to the same component of G then they belong to the same component of H.* ▮

In particular every enclosed subgraph of a connected graph is connected.



a

Both $[\{a, b, c\}]$ and $[\{a, b, c, d\}]$ are enclosed subgraphs

b

$[\{a, b, c\}]$ is not an enclosed subgraph

Fig. 3

Lemma 2. *Let H be an enclosed subgraph of a graph G. Let a, b be distinct vertices of H and let P be a path of G joining a to b. If there is a path Q of H joining a to b such that (i) $P \cap Q = \{a, b\}$ and (ii) $l(Q) \geqq l(P)$, then $P \subseteqq V(H)$.*

Proof. We proceed by induction on $l(P)$. If $l(P)=1$ the claim is obvious. Let $P=(a=p_0, p_1, p_2, \ldots, p_n=b)$, where $m>1$, let $Q=(a=q_0, q_1, q_2, \ldots, q_k=b)$, and suppose that $P \nsubseteq V(H)$. Then, as H is enclosed and $D_H(a, b) \geqq l(P)$ it follows, from (ii), that $\{a, b\} \neq P \cap V(H)$. All vertices of $P \cup Q$ lie in the same component of G so, there is a vertex $p_j \in (P \setminus \{a, b\}) \cap V(H)$ and a path P' in H joining p_j to some q_i and satisfying $P \cap P' = \{p_j\}$, $Q \cap P' = \{q_i\}$. Let $P_1 = \{p_0, p_1, \ldots, p_i\}$ and let $P_2 = \{p_i, p_{i+1}, \ldots, p_m\}$. If $l(P_1) \leqq l(P') + i = l(P' \cup \{q_0, q_1, q_2, \ldots, q_i\})$ then, by the induction hypothesis, $P_1 \subseteqq V(H)$. Then $l(P_2) \leqq l(Q \cup \{p_0, p_1, p_2, \ldots, p_j\})$ and so, again by the induction hypothesis, $P_2 \subseteqq V(H)$. Otherwise, $l(P_1) > l(P') + i$. Since $l(P_1) + l(P_2) \leqq l(Q) = k$ it follows that $l(P_2) < k - i$. Finally, applying the argument above to P_2 first and then to P_1 gives $P_2 \subseteqq V(H)$ and $P_1 \subseteqq V(H)$, so $P \subseteqq V(H)$. ▮

Lemma 3. *Every enclosed subgraph of a graph is isometric.*

Proof. Let H be an enclosed subgraph of a graph G and let a, b be distinct vertices of H. Let P be a shortest path in G joining a to b, and let $Q=(a=c_0, c_1, c_2, \ldots, c_n=b)$ be a path in H joining a to b. Let $P \cap Q = \{c_{i_0}, c_{i_1}, c_{i_2}, \ldots, c_{i_k}\}$ where $0=i_0<i_1< <i_2<\ldots<i_k=n$. Let P_{i_j} (Q_{i_j}) denote the path consisting of the vertices of $P(Q)$ joining c_{i_j} to $c_{i_{j+1}}$, for each $j=0, 1, 2, \ldots, k-1$. Evidently, $l(P_{i_j}) \leqq l(Q_{i_j})$ for otherwise, $P' = (P \setminus P_{i_j}) \cap Q_{i_j}$ is a path joining a, b and $l(P') < l(P)$. By Lemma 2, it follows that $P_{i_j} \subseteqq V(H)$ for each $j=0, 1, 2, \ldots, k-1$, whence $P \subseteqq V(H)$. ▮

Another concept which is useful in the study of enclosed sets is that of a "partial enclosure". Let H be a subgraph of G. Set H^0 to be that subgraph of G whose vertex set consists of the vertices of H and all those vertices c of G for which there are distinct vertices a, b of H and a path P, containing c, joining a to b, and such that $l(P) = d_G(a, b)$. (Note that if the vertices a, b of H^0 belong to the same component of G then they belong to the same component of H^0.) For an integer $n>0$ let $V(H^n)$ denote the collection of all vertices c of G for which there exist distinct vertices a, b

of $V(H^{n-1})$ and for which there is a path P joing a to b and containing c satisfying

$$P \cap V(H^{n-1}) = \{a, b\}$$

and

$$D_{H^{n-1}}(a, b) \geqq l(P).$$

Set $H^n = [V(H^n)]$ — the n^{th} *partial enclosure*. Since every component of G intersects H^{n-1} in a connected subgraph, this inequality amounts to saying $l(P) \leqq l(Q)$ for some path Q in H^{n-1} joining a to b.

For further reference, we set $\overline{H} = [\bigcup_{n \in \mathbf{N}} V(H^n)]$. For completeness sake note that an isolated vertex v of G is in $V(H^0)$ if $v \in V(H)$, but $v \notin V(H^1)$. Therefore, $V(H^{n-1}) \subseteqq V(H^n)$ unless $n=1$ in which case $V(H^1)$ contains all non-isolated vertices from $V(H^0)$.

Lemma 4. *Let H be a subgraph of a graph G. Then the subgraph $\overline{H}$ is an enclosed subgraph.*

Proof. Let a, b distinct vertices of $\overline{H}$ and let P be a path joining a to b which satisfies $P \cap V(\overline{H}) = \{a, b\}$. If $l(P) \leqq D_{\overline{H}}(a, b)$ then there is a path $Q \subseteqq V(\overline{H})$ from a to b such that $l(Q) \geqq l(P)$. Since Q is finite it follows that $Q \subseteqq V(H^n)$, for some n, and hence $P \subseteqq V(H^{n+1})$. Therefore, $P=(a, b)$. In any case, $P \subseteqq V(\overline{H})$, whence $\overline{H}$ is an enclosed subgraph. ∎

The next result shows that partial enclosures of a set are, in some sense, bounded.

Lemma 5. *Let G be a graph. If H is a subgraph of K and K is an enclosed subgraph of G, then $\overline{H}$ is also a subgraph of K.*

Proof. We show that $V(H^n) \subseteqq V(K)$ for each $n \in \mathbf{N}$. For $n=0$, we have by assumption that $V(H^0) \subseteqq V(K)$. Suppose $V(H^{n-1}) \subseteqq V(K)$. Let $c \in V(H^n) \setminus V(H^{n-1})$. Then there are distinct vertices $a, b \in V(H^{n-1})$ and paths P and Q joining a to b such that $c \in P$, $P \cap V(H^{n-1}) = \{a, b\}$, $Q \subseteqq V(H^{n-1}) \subseteqq V(K)$ and $l(Q) \geqq l(P)$. Since $Q \subseteqq V(K)$ and K is an enclosed subgraph, $P \subseteqq V(K)$ by Lemma 2. Hence $V(H^n) \subseteqq V(K)$ for each $n \in \mathbf{N}$, so $V(\overline{H}) \subseteqq V(K)$. ∎

Lemma 6. *Let H be a subgraph of a graph G. Then*
$\overline{H} = [\bigcap (U | V(H) \subseteqq U \subseteqq V(G)$ *and* $[U]$ *is an enclosed subgraph of* $G)]$.

Proof. Let H' denote *dexter*.

As $V(H) \subseteqq V(\overline{H}) \subseteqq V(G)$ and since $\overline{H}$ is an enclosed subgraph (Lemma 4) is follows that $V(H') \subseteqq V(\overline{H})$.

If U satisfies $V(H) \subseteqq U \subseteqq V(G)$ and if $[U]$ is an enclosed subgraph of G then from Lemma 5 we have that $V(\overline{H}) \subseteqq U$ and so $V(\overline{H}) \subseteqq V(H')$ whence $\overline{H} = H'$. ∎

In summary, for each subgraph H of a graph G, $\overline{H} = [\bigcup_{n \in \mathbf{N}} V(H^n)] = [\bigcap (U | V(H) \subseteqq U \subseteqq V(G)$ and $[U]$ is an enclosed subgraph of $G)]$.

The next proposition shows that the prefix on "enclosed" is unnecessary.

Proposition 7. *Let G be a graph. The operator which assigns to each subgraph H the subgraph $\overline{H}$, is a closure operator.*

Proof. It is clear that $\overline{\emptyset}=\emptyset$. Let H and K be subgraphs of G. Then $H\subseteq\overline{H}$ and $K\subseteq\overline{K}$. If $H\subseteq K$ then $H\subseteq\overline{K}$ whence by Lemma 5, $\overline{H}\subseteq\overline{K}$. Finally $\overline{H}\subseteq\overline{\overline{H}}$. Since $\overline{H}$ is an enclosed subgraph containing $V(\overline{H})$ it follows that

$$\overline{\overline{H}}=[\bigcap(U|V(\overline{H})\subseteq U\subseteq V(G) \text{ and } [U] \text{ is an enclosed subgraph of } G)]\subseteq\overline{H}. \quad \blacksquare$$

It is well known that the closed sets defined by a closure operator, when ordered by set inclusion form a complete lattice. The lattice of enclosed subgraphs of a graph G is hereafter referred to as $L(G)$.

Recall that in a lattice L, $c\in L$ is *compact* if whenever $c\leqq\bigvee S$ there exists a finite subset $S'\subseteq S$ with $c\leqq\bigvee S'$. A complete lattice L is said to be *algebraic* if every element of L is the supremum of a set of compact elements.

Proposition 8. *The lattice of enclosed subgraphs of a graph is algebraic.*

Proof. Let G be a graph. For any enclosed subgraph H of G, in the lattice we have $H=\bigvee_{\alpha\in H}\{a\}$. Therefore to complete the proof it is only necessary to show that the atoms of $L(G)$ are compact.

Let $(H_\alpha)_{\alpha\in I}$ be a collection of enclosed subgraphs and suppose

$$\{a\}\leqq\bigvee_{\alpha\in I}H_\alpha. \quad \text{Since} \quad \bigvee_{\alpha\in I}H_\alpha=[\overline{\bigcup_{\alpha\in I}V(H_\alpha)}]=[\bigcup_n(\bigcup_{\alpha\in I}V(H_\alpha))^n], \text{ let } F=\bigcup_{\alpha\in I}V(H_\alpha).$$

We show, by induction on n, that if $\alpha\in F^n$ then there is a finite subset J of I such that $\{a\}\leqq\bigvee_{\alpha\in J}H_\alpha$. If $n=0$ then either $a\in F$ in which case $a\leqq H_\alpha$ for some α or, there exist distinct vertices $b,c\in F$, say $b\in V(H_{\alpha_1})$, $c\in V(H_{\alpha_2})$, and a path P joining b to c, containing a and such that $l(P)=d_G(b,c)$. In this case $a\in\overline{H_{\alpha_1}\cup H_{\alpha_2}}$ so $\{a\}\leqq\leqq H_{\alpha_1}\vee H_{\alpha_2}$. In general, if $a\in F^n\setminus F^{n-1}$ then there exist distinct $b,c\in V(F^{n-1})$ and also paths P and Q both joining b to c with $a\in P$, $Q\subseteq V(F^{n-1})$, $P\cap Q=\{b,c\}$ and $l(Q)\geqq l(P)$. From the induction hypothesis it follows that for each $q\in Q$ there is a finite subset J_q of I such that $\{q\}\leqq\bigvee_{\alpha\in J_q}H_\alpha$. Let $J=\bigcup_{q\in Q}J_q$. As Q is finite, J is a finite subset of I and $a\in[\overline{\bigcup_{\alpha\in J}V(H_\alpha)}]=\bigvee_{\alpha\in J}H_\alpha$. ∎

We will study the lattice $L(G)$ in more detail in the next section. To complete this section, we return to our motivation and prove

Proposition 9. *Every enclosed subgraph of a reflexive graph G is a retract.*

Proof. The proof relies on this result from [6, Theorem 2, p. 341]: *in a reflexive graph every isometric tree is a retract.*

Let H be an enclosed subgraph of a reflexive graph G. At the outset let us suppose in addition that H is connected. Let T be a spanning tree of H. The following artifice is helpful: we associate with the reflexive graph G another graph G^* whose vertex set is $V(G)$ and (a,b) is an edge just if $(a,b)\in E(G)$ and one or both a,b lies in $V(G)\setminus V(H)$ or if (a,b) is an edge of T. Since H is an enclosed subgraph of G, T is an isometric tree of G^* and so by the result cited above, T is a retract of G^* under a retraction map f. Since $f(a)=a$ for all $a\in T$, f applied to G is also a retraction map of G onto H. Finally, if H is not connected then we apply this argument to each of the components of G which contain a component of G. The remaining components of G can be mapped arbitrarily. ▮

Remark. The proof of Proposition 9 will also show that a subgraph H of a reflexive graph G is a retract of G provided that it satisfies the dollowing conditions (cf. the definition of *enclosed subgraph*): for each $a \neq b$ in $V(H)$ and for each path P of G joining a to b, if $P \cap V(H) = \{a, b\}$ then $l(P) \geqq D_H(a, b)$.

However, the collection of all *such* subgraphs of G, ordered by inclusion, does not form a lattice.

Examples. The following diagrams offer some insight into the lattice structure of the set of enclosed subgraphs of a graph.

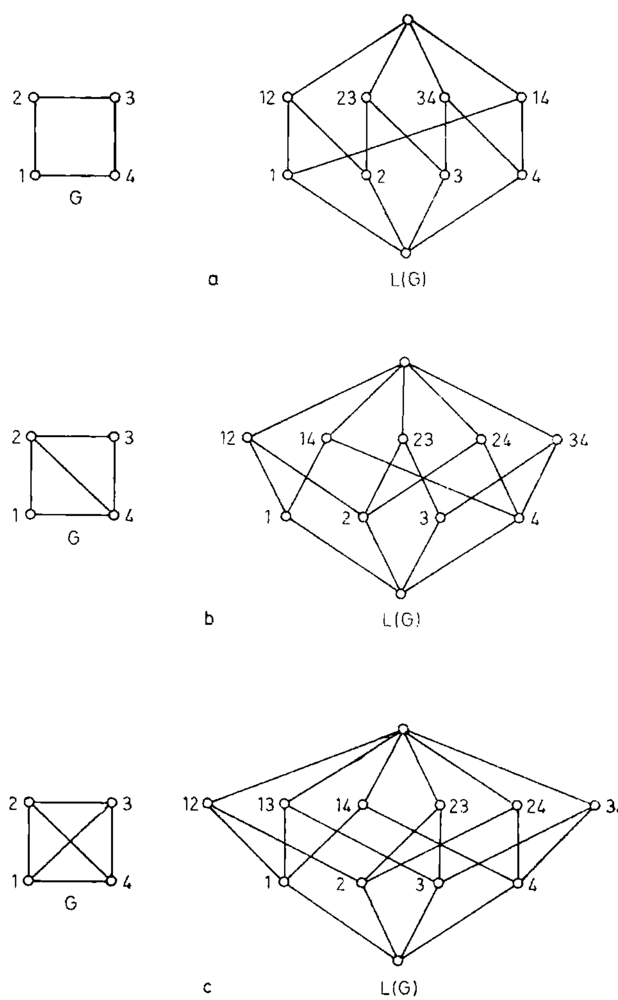


Fig. 4

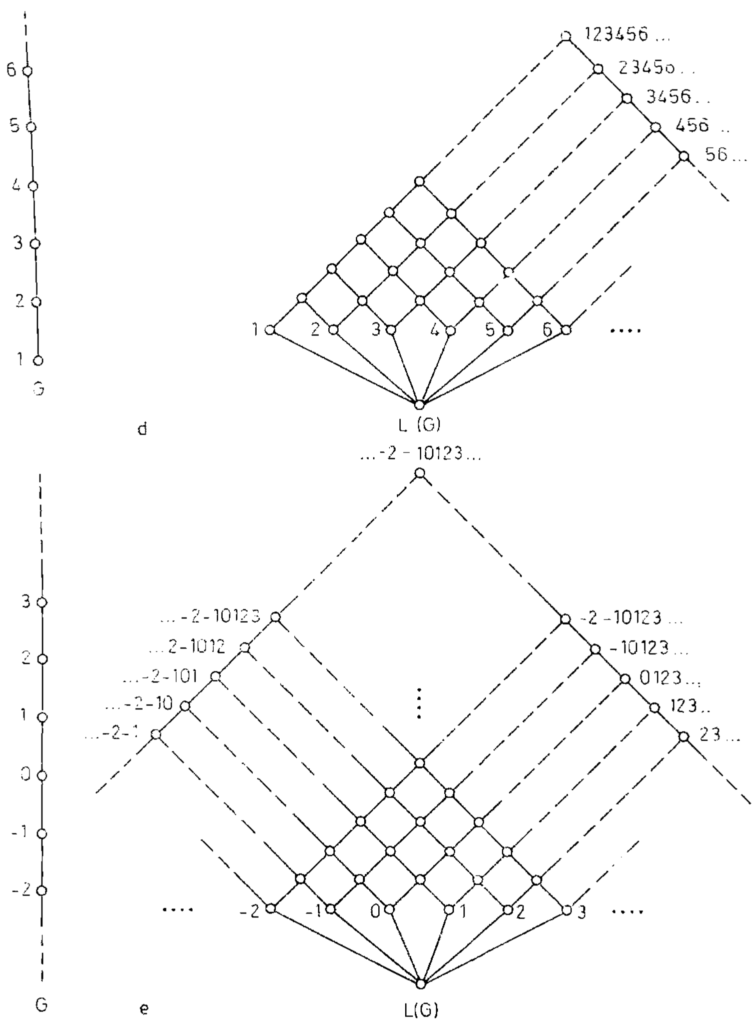


Fig. 5 (continued)
$L(G)$ is isomorphic to the lattice of closed intervals of the integers ordered by inclusion

Meet Representations

The purpose of this section is to prove

Theorem 10. *In the complete lattice of enclosed subgraphs of a reflexive graph, every element is the infimum of an irredundant family of infimum irreducible elements.*

A lattice L is *strongly atomic* if, for every $a<b$ in L, the interval sublattice $[a, b]=\{x\in L|a\leqq x\leqq b\}$ contains an element c such that $a<y\leqq c$ implies $y=c$ for

each $y \in L$. (Such an element c is also called an *upper cover* of a or an *atom* of $[a, b]$.) In [1] (cf. [2]) P. Crawley proved that *in a strongly atomic algebraic lattice every element is the infimum of an irredundant family of infimum irreducible elements.* Therefore, in the light of Proposition 8, once we prove that $L(G)$ is strongly atomic we have established Theorem 10. The substance of the proof will lie in examining the case in which $H \prec F$ in $L(G)$ and H is *connected.* To this end let us suppose that H is a connected, enclosed subgraph of G, that $H \prec F$ in $L(G)$, and that the sublattice $[H, F]$ contains no atom. It follows that there is a sequence $a_0, a_1, a_2, \ldots$ of vertices of G such that

$$F > H_0 > H_1 > H_2 > \ldots > H$$

where $H_i = H \vee \{a_i\}$ and a_i is adjacent to a vertex b_i of H for each $i = 0, 1, 2, \ldots$. We shall show that every such sequence must be finite, whence, H has an upper cover in $[H, F]$. To see this we refer to this result concerning paths and partial enclosures:

Proposition 11. *Let H be an enclosed subgraph of a graph G. Let a, a' be distinct vertices of G not in H and let b, b' be vertices of H. If $(a, b), (a', b') \in E(G)$ and $a' \in \overline{H \cup \{a\}}$ then there are disjoint paths P, Q which either join a to a' and b to b', respectively, or join a to b' and a' to b, respectively. Moreover, $P \cup Q \subseteq V(\overline{H \cup \{a\}})$.*

Let $i > 0$. According to Proposition 11 there is a cycle C_0 in H_0 which contains the vertices a_0, b_0, a_i, b_i. Now, if *in* H_i there were arbitrarily long paths joining a_i to b_i then it would follow that $C_0 \subseteq V(H_i)$ and, in particular, $a_0 \in V(H_i)$ so $H_0 \leqq H_i$. Therefore, $D_{H_i}(a_i, b_i)$ is finite. Let $k_0 = |C_0|$ and set $k_i = D_{H_i}(a_i, b_i)$ for each $i > 0$. In fact, $k_0 > k_i$ for each $i > 0$. Again, since $H_i \not\leqq H_{i+1}$ it follows that $k_0 > k_1 > k_2 \ldots$. There can then be only finitely many a_i's.

We now treat the case that H and F are enclosed subgraphs of G, that $H \prec F$, and that H is not connected. Then H is the disjoint union of a family $(H_i | i \in I)$ of connected subgraphs and so F is also the disjoint union of a family $(F_j | j \in J)$ of connected subgraphs, where I, J are ordinals and $I \subseteq J$. If $I \neq J$ then there is a component F_j of F with $j \notin I$. If we choose any $a \in F_j$ then $H \vee \{a\}$ is an upper cover of H in $[H, F]$. Therefore, we may assume that $I = J$ and as $H \neq F$ there is $i \in I$ such that $H_i \neq F_i$. As H_i is connected there is a vertex a in F_i and not in H_i for which $H_i \vee \{a\}$ is an upper cover of H_i in $[H_i, F_i]$. It is now a simple matter to verify that $H \vee \{a\}$ is an upper cover of H in $[H, F]$. This completes the proof of Theorem 10.

It remains then to verify Proposition 11 which, in turn, relies on this pair of lemmas.

Lemma 12. *Let r, s, t, u, be vertices of a graph G and let P, Q, R, be paths of G which join s to r, s to t, and u to t respectively. If $s \notin R$ and $P \cap Q = \{s\}$ then there are disjoint paths S, T which either join s to r and u to t, respectively, or join u to r and s to t, respectively. Moreover, $S \cup T \subseteq P \cup Q \cup R$.*

Proof. Let $R = (u = r_0, r_1, r_2, \ldots, r_k = t)$. Let i be the least index such that r_i is in $P \cup Q$. If $r_i \in P$ then Q and the path consisting of $u = r_0, r_1, r_2, \ldots, r_i$ augmented by that segment of P from r_i to r are the required paths; if $r_i \in Q$ then choose P and the path consisting of $u = r_0, r_1, r_2, \ldots, r_i$ augmented by that segment of Q from r_i to t. ∎

Lemma 13. *Let H be an enclosed subgraph of a graph G. Let a, b, c be vertices of G satisfying the following conditions: (i) $a \notin V(H)$; (ii) $b \in V(H)$; (iii) $(a, b) \in E(G)$; $c \in V(\overline{H \cup \{a\}}) \setminus V(H \cup \{a\})$. Then there is a cycle C in $\overline{H \cup \{a\}}$ which contains a, b, c.*

Proof. Let $A = H \cup \{a\}$ and let $n > 0$ be that integer for which $c \in V(A^n) \setminus V(A^{n-1})$. We show by induction on n that there are paths P and Q joining c to a and c to b, respectively, such that $P \cup Q \subseteq V(A^n)$ and $P \cap Q = \{c\}$. The union of these paths (possibly with the edge (a, b)) is the required cycle.

According to the definition of the n^{th} partial enclosure there are distinct vertices $x, y \in A^{n-1}$ and paths U, V both joining x to y such that $c \in U$, $U \cap A^{n-1} = \{x, y\} = U \cap V$ and $V \subseteq V(A^{n-1})$.

Let $n = 1$. Suppose $x = a$. If $b \in U \cup V$ then we are ready; if $b \notin U \cup V$ then there is $b' \in V$ satisfying $(a, b') \in E(G)$. As H is enclosed and $a \notin V(H)$ it follows that $(b, b') \in E(G)$. We can now extend V to V' by inserting b between a and b' in V. Then $U \cup V'$ constitutes the required cycle. The case $y = a$ is symmetric . Now suppose that $x \neq a \neq y$. Still, $a \in V$. Let b', b'' be distinct vertices of V both adjacent to a. If $b' \neq b \neq b''$ then, since H is enclosed, (b, b'), (b, b''), $(b', b'') \in E(G)$, whence $a \in V(H)$. Therefore, $b \in V$ and $U \cup V$ is the required cycle.

Let $n > 1$. Suppose that $x, y \notin V(H)$. Then either $x \neq a$ or $y \neq a$. Say $x \neq a$. As $x, y \in V(A^{n-1})$ we have by the induction hypothesis that there are paths P_x, Q_x, P_y and Q_y joining x to a, x to b, y to a, and, y to b, respectively, and satisfying $P_x \cup Q_x \cup P_y \cup Q_y \subseteq V(A^{n-1})$, and $P_x \cap Q_x = \{x\}$, $P_y \cap Q_y = \{y\}$. Now, either $x \notin P_y$ or $x \notin Q_y$, say, $x \notin P_y$. Setting $r = a, s = x$, $t = b$, $u = y$, $P = P_x$, $Q = Q_x$ and $R = P_y$ in Lemma 11, we obtain disjoint paths S, T which either join x to a and b to y, respectively, or join x to b and a to y, respectively. (Note that $S \cup T \subseteq V(A^{n-1})$ but $U \cap V(A^{n-1}) = \{x, y\}$.) In either case, $U \cup S \cup T$ is the desired cycle. If $x \in P_y$ but $x \notin Q_y$ then we can simply interchange the roles of a and b above.

Suppose that $x \in V(H)$ and $y \notin V(H)$. This time apply Lemma 11 by setting $r = a$, $s = y$, $t = b$, $u = x$, $P = P_y$, $Q = Q_y$, where P_y, Q_y are the paths guaranteed by the induction hypothesis in $V(A^{n-1})$ from y to a, y to b, repectively, with $P_y \cap Q_y = \{y\}$, and let R be a path in H joining x to b. (Such a path R exists since x and b are connected in G, and H is an enclosed subgraph.) Then the paths S, T provided by Lemma 11 and augmented by U yield the required cycle. The case $x \notin V(H)$ and $y \in V(H)$ is symmetric to this one.

Finally, even if $x, y \in V(H)$ at least one vertex c', say, of V lies outside of H (since $c \notin V(H)$ and $H = \overline{H}$). We may now set $y = c'$ in the case above to obtain an appropriate cycle containing a, b, and c. ▮

With these lemmas the proof of Proposition 11 is straightforward. Indeed, in view of Lemma 13, there are paths P and Q joining a' to a and a' to b, respectively, with $P \cup Q \subseteq V(\overline{H \cup \{a\}})$. Since b and b' are in the same component of G and $H = \overline{H}$ there is a path R in H joining b to b' (so $a' \notin R$). Applying Lemma 12 with $r = a$, $s = a'$, $t = b$, and $u = b'$ we obtain the required paths. ▮ ▮

Generators of a Graph

Let G be a graph and let k be a cardinal. We say that G is *k-generated* if there are vertices $a_1, a_2, \ldots, a_k$ of G such that $G = \bigvee_{i=1}^{k} \{a_i\}$ in $L(G)$. The set of vertices $\{a_i | i=1, 2, \ldots, k\}$ is called a *generating* subset of G and the vertices a_i are called *generators*.

Theorem 14. *Every graph can be embedded in a 2-generated graph.*

Proof. Let A be a generating subset of G. Suppose that the diameter d of A is finite. Fix a vertex $a \in A$ and let v be an element, $v \notin V(G)$. Now construct a graph G' from G by associating with each vertex $x \in A$ a new path $P(v, x)$ from v to x of length $d+1-k$ where $k = d_G(a, x)$. This new graph G' contains G as a subgraph and G' is 2-generated. Finally, if the diameter d of A is infinite (or if G is disconnected) first adjoin to G a new vertex v_1 adjacent to each vertex of G. Then embed this graph in a 2-generated graph by first selecting a generating subset containing v_1. ∎

The graph illustrated in Fig. 4(e) is not finitely generated but it can be embedded in a 2-generated graph simply by adjoining two more vertices (see Fig. 5).

Let G be a graph and let A be a generating subset of G. We say that A is an *irredundant* generating subset if $\overline{A \setminus \{a\}} \neq G$ for each $a \in A$. The graph of Figure 4(e), for instance, has no irredundant generating subset.

A *coatom* in a lattice L with greatest element is a lower cover of the greatest element. Let $C(L) = \{c \in L | c \text{ coatom of } L\}$.

Theorem 15. *If G is a graph with an irredundant generating subset A then $|A| \geqq |C(L)|$.*

Proof. For each $a \in A$ set $G_a = \overline{A \setminus \{a\}}$. As A is irredundant $G_a < G$ in $L(G)$ for each $a \in A$. Moreover, $G_a \nleqq G_{a'}$ for each $a \neq a'$ in A, and, since $\bar{A} = G$ every coatom

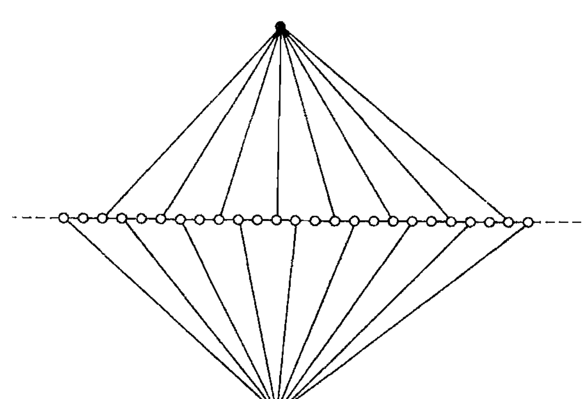


Fig. 5

in $L(G)$ comparable with G_a is not comparable with $G_{a'}$. Now, if for some G_a there is an unbounded increasing sequence $G_a=G_0<G_1<G_2<\dots$ whose supremum in $L(G)$ is G then $\{a\}\leqq \bigvee_{i\in I} G_i=G$ implies that $\{a\}\leqq \bigvee_{i\in I'} G_i$ for some finite $I'\subseteqq I$, since a is compact in $L(G)$ (cf. Proposition 8). This is, of course, impossible since $\bigvee_{i\in I'} G_i=G_{i_0}$ for some $i_0\in I'$ and $\{a\}\nleqq G_i$ for each $i\in I$. ∎

References

[1] P. Crawley, Decomposition theory for nonsemimodular lattices, *Trans. Amer. Math. Soc.* **99** (1961), 246—254.
[2] P. Crawley and R. P. Dilworth, *Algebraic Theory of Lattices,* Prentice-Hall, Englewood Cliffs, N. J., 1973.
[3] P. Hell, Retracts in graphs, Springer-Verlag Lecture Notes in Math. **406** (1974), pp. 291—301.
[4] P. Hell, Absolute planar retracts and the Four Colour Conjecture, *J. Combin. Theory* B, **17** (1974), 5—10.
[5] P. Hell, Graph retractions, *Atti dei convegni lincei* **17** (1976), Vol. II, 263—268.
[6] R. Nowakowski and I. Rival, Fixed-edge theorem for graphs with loops, *J. Graph Theory* 3 (1979), 339—350.

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